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An approach to a dynamic theory of dynamo action in a rotating conducting fluid

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Dynamo action associated with the motion generated by a random body force $\mathbf{f}(\mathbf{x}, t)$ in a conducting fluid rotating with uniform angular velocity $\mathbf{\Omega}$ is considered. It is supposed that, in the Fourier decomposition of $\mathbf{f}$, only waves having a phase velocity $\mathbf{V}$ satisfying $\mathbf{V}.\mathbf{\Omega} \geqslant 0$ are present and that the Fourier amplitudes of $\mathbf{f}$ are isotropically distributed. The resulting velocity field then lacks reflexional symmetry, and energy is transferred to a magnetic field $\mathbf{h}_0(\mathbf{x}, t)$ provided the scale L of $\mathbf{h}_0$ is sufficiently large. Attention is focused on a particular distribution of $\mathbf{h}_0(\mathbf{x}, t)$ (a circularly polarized wave) for which this dynamo action is most efficient. Under these conditions, the mean stresses acting on the fluid are irrotational and no mean flow develops. It is supposed that

$$\lambda \ll \Omega l^2, \quad \lambda \ll h_0 l \quad \text{and} \quad \nu/\lambda = O(1) \quad \text{or smaller},$$

where l ($\ll L$) is the scale of the $\mathbf{f}$-field, and ν and λ are the kinematic viscosity and magnetic diffusivity of the fluid. The response to $\mathbf{f}$ is then dominated by resonant contributions near the natural frequencies of the free undamped system. As $\mathbf{h}_0$ grows in strength, these frequencies change, and the dynamo process is rendered less efficient. Ultimately the magnetic energy M (and also the kinetic energy E) asymptote to steady values. Expressions for these values are obtained for the particular situation when $\nu \ll \lambda$ and when the frequency ω_0 characteristic of the $\mathbf{f}$-field is small compared with other relevant frequencies, notably Ω and h_0/l; under these conditions, it is shown that

$$\frac{M}{E} \sim C\left(\frac{\Omega}{\omega_0}\right)^{\frac{1}{2}}\left(\frac{\nu}{\lambda}\right)^{\frac{1}{2}}\frac{L}{l},$$

where C is a number of order unity determined by the spectral properties of the $\mathbf{f}$-field. The implications for the terrestrial dynamo are discussed.

1. Introduction

In two previous papers (Moffatt 1970*a*, *b*, hereafter referred to as I, II) a theory of dynamo action due to a random velocity field has been developed. The theory is based on a double-scale analysis in which attention is concentrated on the development of the magnetic field on a scale L much greater than the scale l of the velocity fluctuations. In this respect, the theory follows the approach developed by Steenbeck, Krause & Radler (1966), Steenbeck & Krause (1967*a*, *b*) and Radler (1968)†. Parallel developments have been reported by Parker (1970, 1971*a–d*); and related studies of periodic dynamos have been presented by Childress (1970) and G. O. Roberts (1970). Further references are given in a recent review of the subject by P. H. Roberts (1971).

In paper I, the basic conditions for dynamo action were derived, and in particular it was shown that, when the velocity field is weak, dynamo action will certainly occur provided the statistical properties of the velocity field lack reflexional symmetry. In paper II it was shown that this condition can be satisfied by a random superposition of inertial waves in a rotating fluid; and a detailed investigation of the back reaction of the growing Lorentz force on the fluid motion was carried out. It was shown that, as might be expected from general energy principles, the effect of the Lorentz force is to limit the growth of magnetic energy density M. In the (assumed) absence of any driving mechanism, M increases to a maximum value small compared with the initial kinetic energy density E_0 of the fluid motion, and ultimately decays to zero as a result of ohmic dissipation.

It is clear that if there is to be a steady (or statistically steady) dynamo, then there must be some sustained source of energy for the fluid motion to compensate for ohmic (and possibly viscous) dissipation. The purpose of this paper is to show that, if a random driving force $\mathbf{f}(\mathbf{x}, t)$ is introduced into the equation of motion, conditions being otherwise as in paper II, then, provided the length scale L available for the growth of large-scale field fluctuations is sufficiently large, the increase in M continues until M is much greater than the kinetic energy density E and that, in general, a steady state with $M \gg E$ is established. This result is grossly insensitive to the statistical properties of the field $\mathbf{f}(\mathbf{x}, t)$. It does, however, depend upon the assumption that no mean velocity is generated in the fluid; it will turn out in retrospect (see §7) that this assumption is self-consistent in a fluid of infinite extent provided the spectrum tensor of $\mathbf{f}$ is axisymmetric about the direction of $\mathbf{\Omega}$. In the presence of fluid boundaries a mean flow will almost certainly develop, and theory will require appropriate modification.‡

The general ideas behind the development can be very simply explained. The equation for the evolution of the large-scale magnetic field $\mathbf{B}_0 = (\mu\rho)^{\frac{1}{2}}\mathbf{h}_0$ is (in the absence of any mean velocity)

$$\partial\mathbf{h}_0/\partial t = \nabla \wedge \langle\mathbf{u} \wedge \mathbf{h}\rangle + \lambda\nabla^2\mathbf{h}_0, \tag{1.1}$$

where $\mathbf{u}$ and $\mathbf{h}$ are the velocity and magnetic field fluctuations on the scale l, and the angular brackets represent an average over scales large compared with l but small compared with L, or equivalently, an average over an ensemble of realizations of the $\mathbf{f}$-field; λ is the magnetic diffusivity of the fluid and μ and ρ are the magnetic permeability and density, assumed uniform. In both papers I and II, we derived an expression for $\langle\mathbf{u} \wedge \mathbf{h}\rangle$ in the form

$$\langle\mathbf{u} \wedge \mathbf{h}\rangle = \lambda^{-1}A_{ij}h_{0j}, \tag{1.2}$$

† These papers are now available in English (Roberts & Stix 1971).

‡ The method of invoking a random body force is advocated in §5 of the review article by P. H. Roberts (1971), and an interesting discussion is presented of possible back-reaction effects of the type examined in this paper. A simplified treatment of the problem has been presented by Moffatt (1972).

where A_{ij} is a real symmetric tensor determined (in principle) by the statistical properties of the velocity field, which are of course governed by the equation of motion, and are therefore influenced both by the rotation vector $\mathbf{\Omega}$ (via the Coriolis force) and by the local magnetic field $\mathbf{h}_0$ (via the Lorentz force). When the Lorentz force is very weak, A_{ij} is independent of $\mathbf{h}_0$, and substitution of (1.2) into (1.1) gives a linear equation for $\mathbf{h}_0$ with constant coefficients. It was shown in I that equation (1.1) in general admits exponentially growing modes; and in II that, if $\mathbf{\Omega} = (0, 0, \Omega)$, the mode of maximum growth rate is a circularly polarized wave of the form

$$\mathbf{h}_0 = h_0(t)\,(\cos Kz, -\sin Kz, 0), \tag{1.3}$$

where, during the 'linear phase',

$$h_0(t) = h_{00}\,e^{mt}, \quad m = -\lambda K^2 + \alpha_1 K/\lambda, \quad \alpha_1 = h_0^{-2} A_{ij} h_{0i} h_{0j}; \tag{1.4}$$

α_1 is a function of the dimensionless number $Q = \Omega l^2/\lambda$. The mode of maximum growth rate is that for which

$$K = \alpha_1/2\lambda^2, \quad m = \alpha_1^2/4\lambda^3, \tag{1.5}$$

but in fact all modes of the form (1.3) are unstable if $K < \alpha_1/\lambda^2$. Important properties of the field (1.3), which persist through the subsequent nonlinear development, are that

$$\nabla \wedge \mathbf{h}_0 = K\mathbf{h}_0, \quad \nabla \mathbf{h}_0^2 = 0, \quad \mathbf{h}_0 . \mathbf{\Omega} = 0. \tag{1.6}$$

As the field (1.3) grows in strength, its influence on the inertial waves becomes important, and A_{ij} will begin to depend on $\mathbf{h}_0$ as well as on Q; equation (1.1) then becomes nonlinear, and the exponential growth of $\mathbf{h}_0$ begins to level off. If there is no source of energy (as in II), then $M(t) = \frac{1}{2}\mathbf{h}_0^2$ reaches a maximum, as mentioned above, and then falls to zero (ultimately as t^{-1}). However, if a statistically steady source of energy is present (through random body forces) then it is to be expected that $M(t)$ may attain a steady level; the level that it does attain is of course of crucial interest.

When $\mathbf{h}_0 . \mathbf{\Omega} = 0$, and when the spectrum of $\mathbf{f}$ is axisymmetric about the direction of $\mathbf{\Omega}$, symmetry conditions ensure that one of the principal axes of A_{ij} is in the direction of the vector $\mathbf{\Omega} \wedge \mathbf{h}_0$, and the other two lie in the plane determined by $\mathbf{\Omega}$ and $\mathbf{h}_0$; hence we may write

$$A_{ij} = \alpha_1 \frac{h_{0i} h_{0j}}{h_0^2} + \beta_1 \left(\delta_{ij} - \frac{h_{0i} h_{0j}}{h_0^2} - \frac{\Omega_i \Omega_j}{\Omega^2}\right) + \gamma_1 \frac{\Omega_i \Omega_j}{\Omega^2} + \eta_1 \left(\frac{h_{0i}\Omega_j + h_{0j}\Omega_i}{h_0 \Omega}\right). \tag{1.7}\dagger$$

Hence, from (1.2),

$$\lambda \langle \mathbf{u} \wedge \mathbf{h} \rangle = \alpha_1 \mathbf{h}_0 + \eta_1 (h_0/\Omega)\,\mathbf{\Omega}. \tag{1.8}$$

It turns out that the coefficients α_1, β_1, γ_1 and δ_1 depend only on the magnitude of $\mathbf{h}_0$ (and not on its direction in the plane perpendicular to $\mathbf{\Omega}$), and so, if attention is restricted to the single mode of maximum initial growth rate with the properties (1.6), then $\nabla\alpha_1 = \nabla\eta_1 = 0$. From (1.8), we then have

$$\nabla \wedge \langle \mathbf{u} \wedge \mathbf{h} \rangle = \lambda^{-1} \alpha_1 \nabla \wedge \mathbf{h}_0, \tag{1.9}$$

† The term involving η_1 was unjustifiably omitted in II; but it is apparent from (1.9) that this omission was inconsequential as far as subsequent developments in II were concerned.

just as for the linear development phase, but α_1 is now a function of $M = \frac{1}{2}\mathbf{h}_0^2$ (as well as of other parameters not involving $\mathbf{h}_0$).† Equation (1.1) then still admits a solution of the form (1.3), and the equation for $M(t)$ becomes

$$\frac{dM}{dt} = \frac{\alpha_1(M)}{\lambda}KM - \lambda K^2 M. \tag{1.10}$$

It is to be expected that $\alpha_1(M)$ will be a decreasing function of M, and that the ultimate steady level of M, given by solving

$$\alpha_1(M) = \lambda^2 K, \tag{1.11}$$

will therefore be an increasing function of $L = K^{-1}$. The principal aim of this paper is to determine $\alpha_1(M)$, and hence to derive the asymptotic ratio of magnetic to kinetic energy.

The analysis of §§ 2–5 is directed towards obtaining and simplifying expressions for α_1 and for the kinetic energy density E in terms of the statistical properties of the force field, the rotation vector $\mathbf{\Omega}$, the local field strength $\mathbf{h}_0$, and the physical properties of the fluid. In the course of the simplification process, we make various assumptions which are collected together at the end of § 5. The simplified expressions for α_1 and E will be found in equations (5.9) and (5.10); in these formulae, $F_0 = \langle \mathbf{f}^2 \rangle$, ω_0 is a characteristic frequency and $k_0(=l^{-1})$ a characteristic wavenumber in the spectrum of $\mathbf{f}$. The simple conclusions of § 6 are based on these formulae alone, together with the ideas of this introductory section. The large-scale force balance is considered in § 7, and the geophysical implications of the results in § 8.

2. Forced waves in a rotating fluid permeated by a uniform field

Consider a fluid of infinite extent, of magnetic diffusivity λ and kinematic viscosity ν, rotating with uniform angular velocity $\mathbf{\Omega}$, and permeated by a magnetic field $\mathbf{h}_0$ (which may for the moment be regarded as uniform). We shall later restrict attention to the situation $\mathbf{h}_0 . \mathbf{\Omega} = 0$ favoured by the dynamo process described in § 1, but the analysis of §§ 2 and 3 is valid for arbitrary orientation of $\mathbf{h}_0$ relative to $\mathbf{\Omega}$. We suppose that the fluid is subjected to a body force distribution which is a stationary random function of $\mathbf{x}$ and of t. Let $F_0 = \langle \mathbf{f}^2 \rangle$, and let l be a length-scale over which $\mathbf{f}$ varies significantly. We shall suppose that the dissipation in the system is weak; more precisely, we suppose that

$$\nu/\lambda = O(1), \quad Q \equiv \Omega l^2/\lambda \gg 1, \quad J \equiv h_0 l/\lambda \gg 1. \tag{2.1}$$

We make no assumption concerning the magnitude of the ratio

$$J/Q \equiv h_0/\Omega l. \tag{2.2}$$

The conditions (2.1) imply that free waves are weakly damped over a wavelength, and that the dispersion relation for these free waves is strongly anisotropic. We shall suppose further that the velocity field $\mathbf{u}(\mathbf{x}, t)$ that is generated, with r.m.s. value u_0, satisfies

$$R_0 \equiv u_0/\Omega l \ll 1. \tag{2.3}$$

† In paper II, α_1 turned out to be a function of the particular dimensionless combination $S = 2tM(t)/\lambda$.

This implies a restriction on F_0 which will be derived in retrospect (equation (6.6) below).

The condition (2.3) allows us to use the following linearized equations for the forced waves:

$$\partial \mathbf{u}/\partial t + 2\mathbf{\Omega} \wedge \mathbf{u} = -\nabla\chi + \mathbf{h}_0 . \nabla \mathbf{h} + \nu\nabla^2 \mathbf{u} + \mathbf{f}, \tag{2.4}$$

$$\partial \mathbf{h}/\partial t = \mathbf{h}_0 . \nabla \mathbf{u} + \lambda \nabla^2 \mathbf{h}, \tag{2.5}$$

$$\nabla . \mathbf{u} = \nabla . \mathbf{h} = 0, \tag{2.6}$$

where $\rho\chi$ is the sum of fluid pressure and magnetic pressure. We may suppose that any non-solenoidal part of $\mathbf{f}$ is absorbed in χ, so that

$$\nabla . \mathbf{f} = 0. \tag{2.7}$$

The general Fourier decomposition of $\mathbf{f}(\mathbf{x}, t)$ can be expressed in the form

$$\mathbf{f}(\mathbf{x}, t) = \int d^3\mathbf{k} \int_0^\infty d\omega \{\mathbf{A}^+(\mathbf{k}, \omega) \cos(\mathbf{k} . \mathbf{x} - \omega t) + \mathbf{A}^-(\mathbf{k}, \omega) \cos(\mathbf{k} . \mathbf{x} + \omega t) + \mathbf{B}^+(\mathbf{k}, \omega) \sin(\mathbf{k} . \mathbf{x} - \omega t) + \mathbf{B}^-(\mathbf{k}, \omega) \sin(\mathbf{k} . \mathbf{x} + \omega t)\}, \tag{2.8}$$

where the $\mathbf{k}$-integration is over the half-space $\mathbf{k} . \mathbf{\Omega} \geqslant 0$. We know that we can expect dynamo action only if there is a basic asymmetry in the statistics of the $\mathbf{f}$-field about planes perpendicular to $\mathbf{\Omega}$, implying a lack of reflexional symmetry in the externally imposed conditions. In order to maximize this asymmetry (and at the same time to achieve a slight simplification in the analysis) we shall assume that

$$\mathbf{A}^-(\mathbf{k}, \omega) \equiv 0, \quad \mathbf{B}^-(\mathbf{k}, \omega) \equiv 0, \tag{2.9}$$

so that the waves generated by the forcing field all have a phase velocity $\mathbf{V}$ $(= \omega\mathbf{k}/k^2)$ satisfying $\mathbf{V} . \mathbf{\Omega} > 0$. Equation (2.8) may then be written

$$\mathbf{f}(\mathbf{x}, t) = \mathscr{R} \int_{\mathbf{k} . \mathbf{\Omega} > 0} d^3k \int_0^\infty d\omega\, \tilde{\mathbf{f}}(\mathbf{k}, \omega)\, e^{i(\mathbf{k} . \mathbf{x} - \omega t)}, \tag{2.10}$$

where $\tilde{\mathbf{f}}(\mathbf{k}, \omega) = \mathbf{A}^+ - i\mathbf{B}^+$. The fields $\mathbf{u}$, $\mathbf{h}$ and χ then admit a similar Fourier decomposition, and equations (2.4)–(2.7) become

$$-i\omega\tilde{\mathbf{u}} + 2\mathbf{\Omega} \wedge \tilde{\mathbf{u}} = -i\mathbf{k}\tilde{\chi} + ih_{0k}\tilde{\mathbf{h}} - \nu k^2 \tilde{\mathbf{u}} + \tilde{\mathbf{f}}, \tag{2.11}$$

$$-i\omega\tilde{\mathbf{h}} = ih_{0k}\tilde{\mathbf{u}} - \lambda k^2 \tilde{\mathbf{h}}, \tag{2.12}$$

$$\mathbf{k} . \tilde{\mathbf{u}} = \mathbf{k} . \tilde{\mathbf{h}} = \mathbf{k} . \tilde{\mathbf{f}} = 0, \tag{2.13}$$

where $h_{0k} = \mathbf{h}_0 . \mathbf{k}$. From (2.12),

$$\tilde{\mathbf{h}} = -h_{0k}(\omega + i\lambda k^2)^{-1}\tilde{\mathbf{u}}, \tag{2.14}$$

and manipulation of (2.11) and (2.13) then leads to

$$\tilde{\mathbf{u}} = D^{-1}[2(\mathbf{k} . \mathbf{\Omega})\, \mathbf{k} \wedge \tilde{\mathbf{f}} - i\sigma k^2 \tilde{\mathbf{f}}], \tag{2.15}$$

where

$$D = k^2\sigma^2 - 4(\mathbf{k} . \mathbf{\Omega})^2, \quad \sigma = -(\omega + i\nu k^2) + (\omega + i\lambda k^2)^{-1} h_{0k}^2. \tag{2.16}$$

For small λ, and $\nu/\lambda = O(1)$, the expressions for σ and D may be expanded in power series:

$$\sigma(\mathbf{k}, \omega, \lambda) = \sum_{n=0}^{\infty} (i\lambda)^n \sigma_n(\mathbf{k}, \omega), \quad D(\mathbf{k}, \omega, \lambda) = \sum_{n=0}^{\infty} (i\lambda)^n D_n(\mathbf{k}, \omega), \tag{2.17}$$

the coefficients σ_n and D_n being real. Clearly

$$D_0(\mathbf{k},\omega) = k^2\left(\omega - \frac{h_{0k}^2}{\omega}\right)^2 - 4(\mathbf{k}.\boldsymbol{\Omega})^2, \quad D_1(\mathbf{k},\omega) = 2k^4\left(\omega - \frac{h_{0k}^2}{\omega}\right)\left(\frac{h_{0k}^2}{\omega^2} + \frac{\nu}{\lambda}\right). \quad (2.18)$$

When the dissipation is weak, the response to the driving force is, as in any weakly damped linear system, most pronounced when ω is near to one of the natural frequencies, ω_c say, of the undamped system, determined by

$$D_0(\mathbf{k},\omega_c) = 0. \quad (2.19)$$

The two positive roots of (2.19) are

$$\omega_c = (-1)^c\,\Omega\cos\theta + (\Omega^2\cos^2\theta + h_{0k}^2)^{\frac{1}{2}} \quad (c = 1, 2), \quad (2.20)$$

where θ is the angle between $\boldsymbol{\Omega}$ and $\mathbf{k}$ $(0 \leqslant \theta \leqslant \frac{1}{2}\pi)$. These satisfy

$$\omega_c - h_{0k}^2/\omega_c = (-1)^c\,2\Omega\cos\theta. \quad (2.21)$$

The relations (2.20) can be regarded as defining two three-dimensional manifolds $\mathscr{M}_1$ and $\mathscr{M}_2$ in the four-dimensional space of the variables $(\mathbf{k},\omega)$. It may happen that the spectrum of $\mathbf{f}$ is confined to a region $\mathscr{D}$ of this space that does not intersect either $\mathscr{M}_1$ or $\mathscr{M}_2$, in which case there is no possibility of a resonant response to the forcing field. We shall suppose in what follows that there *is* a real intersection of $\mathscr{D}$ with at least one of the resonant manifolds $\mathscr{M}_1$ and $\mathscr{M}_2$ and that the response is dominated by contributions from this intersection.

It may be noted that the free mode associated with the frequency ω_1 has positive helicity (i.e. $(i\mathbf{k}\wedge\tilde{\mathbf{u}}).\tilde{\mathbf{u}}^* > 0$, see II) and that with ω_2 has negative helicity; a *net* helicity is in general generated because the spectral responses determined by (2.15) are in general unequal at these two frequencies.

3. Evaluation of the kinetic energy density and the mean electromotive force

We may reasonably assume that the forcing field $\mathbf{f}(\mathbf{x},t)$ does not exhibit any intrinsic helicity (so that any lack of reflexional symmetry arises solely through the interaction of the $\mathbf{f}$-field and the rotation $\boldsymbol{\Omega}$); to be precise, we assume that the helicity spectrum of $\mathbf{f}$ is identically zero, i.e. that

$$\langle (i\mathbf{k}\wedge\tilde{\mathbf{f}}).\tilde{\mathbf{f}}^*\rangle \equiv 0. \quad (3.1)$$

Moreover, since $\mathbf{k}.\tilde{\mathbf{f}} = 0$, it is clear that

$$\langle|\mathbf{k}\wedge\tilde{\mathbf{f}}|^2\rangle = k^2\langle|\tilde{\mathbf{f}}|^2\rangle = (2\pi)^{-1}F(\mathbf{k},\omega), \text{ say}, \quad (3.2)$$

and so, from (2.15)

$$\langle|\tilde{\mathbf{u}}|^2\rangle = \frac{k^2}{2\pi|D|^2}(4(\mathbf{k}.\boldsymbol{\Omega})^2 + |\sigma|^2k^2)\,F, \quad (3.3)$$

and, from (2.14) and (2.15), after some reduction,

$$\mathscr{R}\langle\tilde{\mathbf{u}}\wedge\tilde{\mathbf{h}}^*\rangle = \frac{2\mathbf{k}h_{0k}\lambda k^4(\mathbf{k}.\boldsymbol{\Omega})\,(\mathscr{R}\sigma)\,F}{\pi|D|^2\,(\omega^2+\lambda^2k^4)}. \quad (3.4)$$

Note that this latter expression vanishes if either λ or $\mathbf{\Omega}$ vanishes; both dissipation and rotation are essential for the production of the key quantity $\langle \mathbf{u} \wedge \mathbf{h} \rangle$. Note also that, when $\mathbf{h}_0$ is weak, $|D|^2$ and $\mathscr{R}\sigma$ become independent of $\mathbf{h}_0$, and the expression (3.4) is then linear in $\mathbf{h}_0$; but when $\mathbf{h}_0$ is strong, the dependence of these quantities (particularly of $|D|^2$) on $\mathbf{h}_0$ can obviously cause strong departures from linearity.

In order to obtain expressions for $\langle \mathbf{u}^2 \rangle$ and $\langle \mathbf{u} \wedge \mathbf{h} \rangle$ we have to integrate (3.3) and (3.4) first over ω, then over $\mathbf{k}$. Under the assumptions of § 2, these integrals will be dominated by contributions from neighbourhoods of the resonant manifolds where ω is given by (2.20), and where $|D|^{-2}$ is sharply peaked. We may reasonably assume that

$$F(\mathbf{k},\omega) = O(\omega^2) \quad \text{as} \quad \omega \to 0, \tag{3.5}$$

so that there is no danger of an important contribution (in the limit $\lambda \to 0$) from the neighbourhood of $\omega = 0$.

Now from (2.17),

$$|D|^2 = D_0^2 + \lambda^2(D_1^2 - 2D_0D_2) + O(\lambda^4), \tag{3.6}$$

so that, near either root ω_c of (2.19),

$$|D|^2 = (\omega-\omega_c)^2 D_{0\omega_c}^2 + \lambda^2 D_{1c}^2 + O(\lambda^3), \tag{3.7}$$

where, from (2.18),

$$D_{0\omega_c} = \left[\frac{\partial}{\partial\omega} D_0(\mathbf{k},\omega)\right]_{\omega=\omega_c} = (-1)^c\, 4k^2\Omega\cos\theta \left(1+\frac{h_{0k}^2}{\omega_c^2}\right), \tag{3.8}$$

and

$$D_{1c} = D_1(\mathbf{k},\omega_c) = (-1)^c\, 4k^4\Omega\cos\theta\left(\frac{h_{0k}^2}{\omega_c^2}+\frac{\nu}{\lambda}\right). \tag{3.9}$$

Hence, standard asymptotic analysis (appendix A)† gives, for $\lambda \to 0$ and $\nu/\lambda = O(1)$,

$$E = \tfrac{1}{2}\langle \mathbf{u}^2\rangle = \tfrac{1}{4}\int \langle |\tilde{\mathbf{u}}|^2\rangle\, d^3\mathbf{k}\, d\omega \sim \frac{1}{16\lambda}\sum_{c=1,2}\int \frac{F(\mathbf{k},\omega_c)\,\omega_c^4\, d^3\mathbf{k}}{k^4(h_{0k}^2+\omega_c^2)\,(h_{0k}^2+\omega_c^2\nu/\lambda)}, \tag{3.10}$$

and

$$\langle \mathbf{u}\wedge\mathbf{h}\rangle_i = \tfrac{1}{2}\mathscr{R}\int \langle \tilde{\mathbf{u}}\wedge\tilde{\mathbf{h}}^*\rangle_i\, d^3\mathbf{k}\, d\omega = \lambda^{-1}A_{ij}k_{0j}, \tag{3.11}$$

where

$$A_{ij} \sim -\frac{\lambda}{16}\sum_{c=1,2}(-1)^c \int \frac{k_i k_j F(\mathbf{k},\omega_c)\,\omega_c^2\, d^3\mathbf{k}}{k^3(h_{0k}^2+\omega_c^2)\,(h_{0k}^2+\omega_c^2\nu/\lambda)}. \tag{3.12}$$

The factor $-(-1)^c$ appears in (3.12) because when $\lambda \to 0$, and $\omega = \omega_c$,

$$\mathscr{R}\sigma \sim -(-1)^c\, 2\Omega\cos\theta. \tag{3.13}$$

4. Further reduction of the expressions for E and $\langle \mathbf{u} \wedge \mathbf{h} \rangle$

The integrals (3.10) and (3.12) are extremely complicated, and it seems desirable to introduce further simplifying assumptions to permit further reduction. To this end, we assume

$$\nu \ll \lambda, \quad \mathbf{h}_0\cdot\mathbf{\Omega} = 0, \quad F(\mathbf{k},\omega) = F(k,\omega). \tag{4.1}$$

† An alternative procedure which leads to the same results (3.10)–(3.12) is to evaluate the residues at the relevant poles of the integrand in the complex-ω plane, where $D(\mathbf{k},\omega) = 0$, and to use contour integration.

Of these assumptions, the first is not unduly restrictive, and is very likely to be satisfied in the geophysical context. The second is justifiable on the grounds that the magnetic mode of maximum initial growth rate satisfies this condition, and the situation with $\mathbf{h}_0$ perpendicular to $\mathbf{\Omega}$ is therefore of particular interest. The third means that we assume that the Fourier amplitudes of the **f**-field are isotropically distributed over the available directions $\mathbf{k}.\mathbf{\Omega} > 0$.

Considering first the expression (3.10), it is evident that the dominant contribution to the integral, under the condition $\nu \ll \lambda$, comes from the region of **k** space where $h_{0k} = \mathbf{h}_0.\mathbf{k} \ll 0$. Physically, the reason is that wave modes for which $h_{0k} = 0$ do not bend the magnetic field lines and therefore do not experience ohmic damping; their amplitudes are limited only by viscosity, and when $\nu \ll \lambda$, these modes make the largest contribution to $\langle \mathbf{u}^2 \rangle$; mathematically the integrand is singular on $h_{0k} = 0$ when $\nu/\lambda \to 0$.

The plane $h_{0k} = 0$ intersects the manifolds $\mathscr{M}_1$ and $\mathscr{M}_2$ where

$$\omega_1 = 0 \quad \text{and} \quad \omega_2 = 2\Omega\cos\theta. \tag{4.2}$$

Hence there is no contribution to the expression (3.10) from the term $c = 1$, and there is a contribution from the term $c = 2$ only if (as will be supposed)

$$F(k,\omega) \not\equiv 0 \quad \text{for} \quad \omega \leqslant 2\Omega. \tag{4.3}$$

With $h_{0k} \approx 0$, (3.10) then becomes

$$E \sim \frac{1}{16\lambda}\int \frac{\omega_2^2 F(k,\omega_2)\, d^3\mathbf{k}}{k^4(h_{0k}^2 + \omega_2^2\nu/\lambda)}. \tag{4.4}$$

It will be convenient to introduce polar angles (θ,ϕ) relative to the direction of $\mathbf{\Omega}$, and (θ',ϕ') relative to the direction of $\mathbf{h}_0$, so that, with $\mathbf{\Omega}.\mathbf{h}_0 = 0$,

$$\mathbf{\Omega}.\mathbf{k} = \Omega k\cos\theta = \Omega k\sin\theta'\cos\phi', \tag{4.5}$$

$$\mathbf{h}_0.\mathbf{k} = h_0 k\cos\theta' = h_0 k\sin\theta\cos\phi. \tag{4.6}$$

Since we are concerned only with the half-space $\mathbf{\Omega}.\mathbf{k} \geqslant 0$, the relevant range of these angles is

$$\left.\begin{matrix} 0 \leqslant \theta \leqslant \frac{1}{2}\pi \\ 0 \leqslant \phi \leqslant 2\pi \end{matrix}\right\}, \quad \left\{\begin{matrix} 0 \leqslant \theta' \leqslant \pi \\ -\frac{1}{2}\pi \leqslant \phi' \leqslant \frac{1}{2}\pi \end{matrix}\right\}. \tag{4.7}$$

On the plane $h_{0k} = 0$, $\theta' = \frac{1}{2}\pi$, and $\theta = |\phi'|$, so that $\omega_2 = 2\Omega\cos\phi'$. Putting $h_{0k}^2 = h_0^2 k^2\cos^2\theta'$ and $d^3\mathbf{k} = k^2\sin\theta'\, dk\, d\theta'\, d\phi'$, we can integrate (4.4) with respect to θ'; replacing the variable ϕ' by ω_2 (and dropping the suffix 2), we then have, for $\nu/\lambda \to 0$,

$$E \sim \frac{1}{2h_0(\nu\lambda)^{\frac{1}{2}}}\int_0^\infty\int_0^{2\Omega} \frac{\omega F(k,\omega)\, dk\, d\omega}{2\Omega k^3(1-\omega^2/4\Omega^2)^{\frac{1}{2}}}. \tag{4.8}$$

The integrand has an integrable singularity at the upper limit $\omega = 2\Omega$. Note (i) the appearance of the (large) factor $\nu^{-\frac{1}{2}}$ which has its origin in the physical process described above, and (ii) the very simple relation $E \propto h_0^{-1}$, which is plausible in that, for given $F(k,\omega)$, as h_0 increases there is a corresponding increase in the resistance to excitation of those Fourier components of the velocity field that bend magnetic field lines.

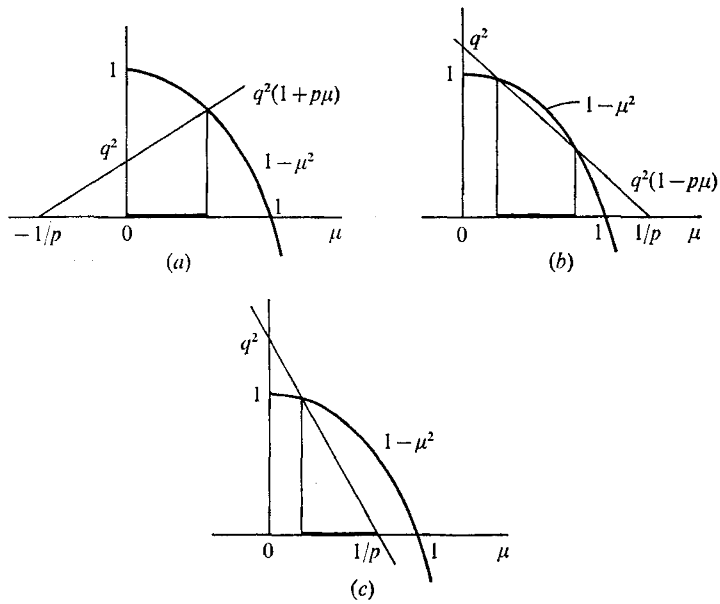

FIGURE 1. The range of integration for the integrals defining $G_c(p,q)$ $(c = 1, 2)$. The appropriate interval of the μ axis is indicated by the thick line. (a) $c = 1, q < 1$; (b) $c = 2$, $p < 1, q^2 > 1$; (c) $c = 2, p > 1, q^2 > 1$.

Turning now to the expression (3.11) for $\langle \mathbf{u} \wedge \mathbf{h} \rangle$ in terms of the tensor A_{ij} given (for small λ) by (3.12), and noting that, for the reasons put forward in §1, when $\mathbf{h}_0 . \mathbf{\Omega} = 0$, we need only evaluate the scalar

$$\alpha_1 = h_0^{-2} A_{ij} h_{0i} h_{0j}, \tag{4.9}$$

we have, from (3.12),

$$\alpha_1 \sim -\frac{\lambda}{16h_0^2} \sum_{c=1,2} (-1)^c \int \frac{h_{0k}^2 F(k, \omega_c)\, \omega_c^2\, d^3\mathbf{k}}{k^3(h_{0k}^2+\omega_c^2)(h_{0k}^2+\omega_c^2 \nu/\lambda)}. \tag{4.10}$$

This quantity, unlike a general component of A_{ij}, and unlike $\langle \mathbf{u}^2 \rangle$, remains finite in the limit $\nu/\lambda \to 0$, and we may simply put $\nu = 0$ in the integrand, i.e.

$$\alpha_1 \sim -\frac{\lambda}{16h_0^2} \sum_{c=1,2} (-1)^c \int\int\int \frac{F(k,\omega_c)\, \omega_c^2 \sin\theta\, dk\, d\theta\, d\phi}{k(h_{0k}^2+\omega_c^2)}, \tag{4.11}$$

and in this expression $\omega_c(k,\theta,\phi)$ is given by (2.19), with $h_{0k} = h_0 k \sin\theta \cos\phi$. If we change variables from (k,θ,ϕ) to (k,μ,ω_c) where $\mu = \cos\theta$, this expression may be reduced (appendix B) to the form

$$\alpha_1 \sim \frac{-\lambda}{32h_0^3} \sum_{c=1,2} (-1)^c \int\int \frac{F(k,\omega_c)}{k^2} G_c\left(\frac{2\Omega}{\omega_c}, \frac{\omega_c}{h_0 k}\right) dk\, d\omega_c, \tag{4.12}$$

with
$$G_c(p,q) = \int \frac{d\mu}{(1 \pm p\mu)^{\frac{1}{2}} (1-\mu^2-q^2(1 \pm p\mu))^{\frac{1}{2}}}, \tag{4.13}$$

where the upper and lower signs correspond here and subsequently to $c = 1, 2$ respectively, and where the range of integration in (4.13) is the subinterval of $[0, 1]$ in which both factors in the integrand are real (see figure 1). (For *given*

k and ω_c, the range of values of μ on the resonant manifolds $\mathscr{M}_c$ is correspondingly restricted by $h_{0k}^2 \leqslant h_0^2 k^2 \sin^2\theta$.) The integrals (4.13) may be expressed in terms of incomplete elliptic integrals of the first kind.

The expression (4.12) is the difference of two integrals, each of which, like (4.8), is a weighted integral over the spectrum function $F(k,\omega)$. It is evident from (4.11) that the two contributions to α_1 corresponding to $c = 1, 2$ have opposite signs, consistent with the observation in the last paragraph of §2, but since $G_1(p,q) \neq G_2(p,q)$ they are of unequal magnitude for general $F(k,\omega)$.

5. Simplification of the expression for α_1 for small excitation frequencies

The expression (4.12) can be simplified further if we assume (consistent with (4.3)) that the spectrum function $F(k,\omega)$ is localized in $(\mathbf{k},\omega)$ space around a characteristic wavenumber k_0 ($= l^{-1}$) and a characteristic frequency ω_0 satisfying

$$\omega_0 \ll \Omega \quad \text{and} \quad \omega_0 \ll h_0 k_0, \tag{5.1}$$

i.e. we assume that typical frequencies in the forcing field are small compared with Ω, and typical phase velocities are small compared with the local Alfvén velocity. It is also helpful to assume, consistent with (5.1), that

$$\omega_0 \ll h_0^2 k_0^2/\Omega. \tag{5.2}$$

Then the dominant contribution to the integral (4.12) cames from the region where $p = 2\Omega/\omega_c \gg 1$ and $q = \omega_c/h_0 k \ll 1$ and we can replace the functions $G_c(p,q)$ in (4.12) by the limiting form as $p \to \infty$ of the functions

$$g_c(p) = G_c(p,0) = \int_0^{\mu_c} \frac{d\mu}{(1 \pm p\mu)^{\frac{1}{2}}(1-\mu^2)^{\frac{1}{2}}}, \tag{5.3}$$

where (appendix B) $\mu_1 = 1$ and $\mu_2 = p^{-1}$. As $p \to \infty$,

$$g_1(p) \sim c_1 p^{-\frac{1}{2}}, \quad g_2(p) = o(p^{-\frac{1}{2}}), \tag{5.4}$$

where $c_1 = 2{\cdot}55\ldots$, and so (4.12) becomes

$$\alpha_1 \sim \frac{c_1 \lambda}{32\sqrt{2}\, h_0^3} \int_0^\infty \int_0^\infty \frac{F(k,\omega)}{k^2} \left(\frac{\omega}{\Omega}\right)^{\frac{1}{2}} dk\, d\omega. \tag{5.5}$$

The corresponding expression for E, from (4.8) with $\omega \ll \Omega$ in the integrand, is

$$E \sim \frac{1}{2h_0(\nu\lambda)^{\frac{1}{2}}} \int_0^\infty \int_0^\infty \frac{\omega}{k^3} F(k,\omega)\, dk\, d\omega. \tag{5.6}$$

The expressions (5.5) and (5.6) have a simple dimensional structure. Noting that

$$F(k,\omega) = \frac{F_0}{k_0 \omega_0} F_1\left(\frac{k}{k_0}, \frac{\omega}{\omega_0}\right), \tag{5.7}$$

where F_1 is a dimensionless function of its arguments, satisfying

$$\int_0^\infty \int_0^\infty F_1(\xi,\eta)\, d\xi\, d\eta = 1, \tag{5.8}$$

we can easily derive from (5.5) and (5.6)

$$\alpha_1 \sim a\lambda l^2 M^{-\frac{3}{4}}(\omega_0/\Omega)^{\frac{1}{2}} F_0, \tag{5.9}$$

and

$$E \sim b(\lambda\nu)^{-\frac{1}{2}} l^3 M^{-\frac{1}{2}}(\omega_0/\Omega) F_0, \tag{5.10}$$

where $l = k_0^{-1}$, $M = \frac{1}{2}\mathbf{h}_0^2$, and a and b are dimensionless constants of order unity defined by

$$a = \frac{c_1}{128}\int_0^\infty\int_0^\infty \eta^{\frac{1}{2}}\xi^{-2}F_1(\xi,\eta)\,d\xi\,d\eta, \quad b = \frac{1}{2\sqrt{2}}\int_0^\infty\int_0^\infty \eta\xi^{-3}F_1(\xi,\eta)\,d\xi\,d\eta. \tag{5.11}$$

It may be helpful to collect together and classify the various assumptions that have been made in arriving at the formulae (5.9) and (5.10); in effect, these are as follows:

(*a*) *Assumptions about the spectrum function* $F(\mathbf{k},\omega)$ *of the force field* $\mathbf{f}$:

(i) $F = F_0 l\omega_0^{-1}F_1(kl, \omega/\omega_0)$, where $F_1(\xi,\eta)$ is localized around $\xi = O(1), \eta = O(1)$; (ii) $\omega_0 \ll \Omega$; (iii) $F = O(\omega^2)$ as $\omega \to 0$; (iv) $F \equiv 0$ for $(\mathbf{k}.\mathbf{\Omega})\,\omega < 0$ (giving a bias to the characteristics of the force distribution in the direction of $\mathbf{\Omega}$).

(*b*) *Assumptions about the diffusion parameters* λ, ν:

(v) $\lambda \ll \Omega l^2$; (vi) $\nu \ll \lambda$.

(*c*) *Assumptions about the large-scale magnetic field*:

(vii) $\mathbf{h}_0.\mathbf{\Omega} = 0$; (viii) $h_0 \gg \lambda/l$; (ix) $h_0 \gg l(\omega_0\Omega)^{\frac{1}{2}}$.

For a field which grows from an infinitesimal level, we obviously should not use the formulae (5.9) and (5.10) until the inequalities (viii) and (ix) are satisfied.

6. The ultimate level of magnetic energy density

Substitution of (5.9) in (1.10) gives

$$\frac{dM}{dt} = a\frac{Kl^2}{M^{\frac{3}{4}}}\left(\frac{\omega_0}{\Omega}\right)^{\frac{1}{2}} F_0 - \lambda K^2 M, \tag{6.1}$$

with solution given by

$$M^{\frac{7}{4}} = \frac{al^2}{\lambda K}\left(\frac{\omega_0}{\Omega}\right)^{\frac{1}{2}} F_0(1-e^{-\frac{7}{4}\lambda K^2 t}) + M_0^{\frac{7}{4}}e^{-\frac{7}{4}\lambda K^2 t}, \tag{6.2}$$

where M_0 is the magnetic energy density at some instant $t = 0$ beyond which the assumptions (viii) and (ix) above are satisfied. Hence in a time of order L^2/λ, (where we take $L = K^{-1}$) the magnetic field settles down to a steady level given by

$$M^{\frac{7}{4}} = \frac{al^2L}{\lambda}\left(\frac{\omega_0}{\Omega}\right)^{\frac{1}{2}} F_0. \tag{6.3}$$

For consistency, this must also satisfy (viii) and (ix), or equivalently

$$M^{\frac{3}{2}} \gg \lambda^3/l^3, \quad l^3(\omega_0\Omega)^{\frac{3}{4}}, \tag{6.4}$$

and clearly these can be satisfied by (6.3) provided L is sufficiently large.

From (6.3) and (5.10), the ratio M/E in the ultimate steady state is given by

$$\frac{M}{E} = C\left(\frac{\Omega}{\omega_0}\right)^{\frac{1}{2}}\left(\frac{\nu}{\lambda}\right)^{\frac{1}{2}}\frac{L}{l}, \tag{6.5}$$

where $C = a/b$ is a number of order unity. The large factor $(\Omega/\omega_0)^{\frac{1}{2}}$ will tend to compensate the small factor $(\nu/\lambda)^{\frac{1}{2}}$; a steady state for which $M \gg E$ is inevitable if L/l is sufficiently large.

It is interesting to note from (5.10) and (6.3) that, in the steady state, both E and M are proportional to $F_0^{\frac{2}{3}}$. This is to be contrasted with the simple proportionality $E \propto F_0$ that arises in a completely linear system. The condition (2.3), or equivalently $E \ll \Omega^2 l^2$, now leads to the appropriate restriction on the magnitude of F_0 (for the validity of the linearized wave treatment given in §2), *viz.*

$$\frac{F_0}{\Omega^3\lambda} \ll \left(\frac{L}{l}\right)^{\frac{1}{2}} \left(\frac{\Omega}{\omega_0}\right)^{\frac{5}{4}} \left(\frac{\nu}{\lambda}\right)^{\frac{3}{4}}. \tag{6.6}$$

The inequalities (6.4) and (6.6) are compatible provided

$$\frac{L}{l} \gg \frac{1}{Q^2}\left(\frac{\omega_0}{\Omega}\right)^{\frac{1}{6}} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}}, \quad \left(\frac{\omega_0}{\Omega}\right)^{\frac{9}{2}} \left(\frac{\lambda}{\nu}\right)^{\frac{3}{2}}. \tag{6.7}$$

7. The large-scale force balance

It has been noted in equation (1.6) that, for the mode $\mathbf{h}_0(\mathbf{x}, t)$ of maximum growth rate, to which attention has been confined, $\nabla \wedge \mathbf{h}_0 = K\mathbf{h}_0$, so that the associated Lorentz force vanishes; i.e. the dynamo mechanism favours the growth of a force-free field when this is permitted by the boundary conditions. We must also, however, consider the effect of the force distribution $\partial P_{ij}/\partial x_j$ arising from the Reynolds stress distribution (augmented by the magnetic effect),

$$P_{ij} = -\langle u_i u_j \rangle + \langle h_i h_j \rangle. \tag{7.1}$$

Since $\mathbf{h}_0 = \mathbf{h}_0(z, t)$, all mean quantities such as P_{ij} depend on z and t only. Moreover, when $\mathbf{h}_0 . \mathbf{\Omega} = 0$, and when the spectrum tensor of $\mathbf{f}(\mathbf{x}, t)$ is axisymmetric about the direction of $\mathbf{\Omega}$, (so that for example $\langle \tilde{f}_1 \tilde{f}_3^* \rangle = 0$) it is evident from perusal of the expressions for

$$\langle \tilde{u}_i \tilde{u}_3^* \rangle, \quad \langle \tilde{h}_i \tilde{h}_3^* \rangle, \quad (i = 1, 2), \tag{7.2}$$

in terms of $\tilde{\mathbf{f}}$ (via (2.14) and (2.15)) that each of (7.2) involves only terms of the form $\cos\phi$ or $\sin\phi$ times a function of $\cos^2\phi$, which integrate to zero over the range $(0, 2\pi)$ or ϕ. It follows that

$$P_{13} = P_{23} = 0, \tag{7.3}$$

i.e. $(0, 0, 1)$ is a principal axis of P_{ij}, irrespective of the direction in the x, y plane of $\mathbf{h}_0$. Hence

$$\frac{\partial}{\partial x_i} P_{ij}(z, t) = \frac{\partial}{\partial z} P_{3j}(z, t) = \frac{\partial}{\partial x_j} P_{33}(z, t). \tag{7.4}$$

This force distribution is irrotational and therefore does not generate any mean velocity, consistent with the assumption made at the outset in §1. There is merely an adjustment of the mean pressure distribution.

This convenient result arises only because we neglect the effects of fluid boundaries. In a fluid of bounded extent, it is known that a force-free field without singularities, and without external sources, cannot exist; and it seems inevitable that the Reynolds stress distribution will also in these circumstances contribute a rotational force. The methods of this paper can in principle be extended to include the effects a mean velocity distribution, but the labour involved would be considerable.

8. Discussion

Under the various assumptions summarized at the end of §5, it has been established that a random forcing field $\mathbf{f}(\mathbf{x}, t)$ whose Fourier amplitudes are isotropically distributed over the half-space $\mathbf{k}.\mathbf{\Omega} \geqslant 0$ generates a random velocity field, whose statistical properties lack reflexional symmetry, and which gives rise to dynamo action; that the energy that is pumped into the velocity field is systematically transferred to the magnetic field, until the Lorentz force reacts back upon the structure of the velocity field in such a way as to arrest this process; and that, if the length scale L available for the growth of large-scale magnetic field Fourier components of the type (1.3) is sufficiently large, then the magnetic energy in these components will, in the steady state that is established, greatly exceed the kinetic energy in the background random wave motion. Just how large L must be depends on the scale l and characteristic frequency ω_0 of the force field; when $\nu \ll \lambda$ and $\omega_0 \ll \Omega$, M is large compared with E provided

$$\frac{L}{l} \gg \left(\frac{\omega_0}{\Omega}\right)^{\frac{1}{2}} \left(\frac{\lambda}{\nu}\right)^{\frac{1}{2}}. \tag{8.1}$$

The applicability of these results to processes in the earth's liquid conducting core must be regarded with caution for several reasons. First, as observed earlier the presence of a rigid boundary almost certainly implies a mean flow in the core irrespective of the nature of the energy source. It is widely believed that differential rotation is a strong ingredient of the mean flow and that this is primarily responsible for the generation of toroidal field from poloidal field (see, for example, Hide & Roberts 1961). On the other hand, it has been demonstrated by Krause & Steenbeck (1967) that the α-effect can definitely generate poloidal from toroidal field (essentially through the mechanism described by Parker 1955), and it seems likely that this is a key mechanism in the dynamo process for the earth's field. Presumably, the earth's toroidal field controls the level of this α-effect through the type of mechanism described in this paper, so that the rate of regeneration of poloidal field is just sufficient to compensate for ohmic dissipation.

It is therefore relevant to enquire whether the various approximations made in this paper bear any relation to the conditions that exist in the earth's core (as inferred from seismological, geomagnetic and other data). The condition $\nu \ll \lambda$ is almost certainly satisfied and need not concern us further. Estimates of other dimensional parameters (Hide & Roberts 1961) are:

$$\lambda \approx 3\,\mathrm{m^2\,s^{-1}}, \quad l \approx 10^4\,\mathrm{m}, \quad u_0 \approx 10^{-3}\,\mathrm{m\,s^{-1}},$$
$$\Omega \approx 7\times 10^{-5}\,\mathrm{s^{-1}}, \quad L \approx 3\times 10^6\,\mathrm{m}, \quad h_0 \approx 0{\cdot}4\,\mathrm{m\,s^{-1}}.$$

These choices give

$$Q \approx 2\times 10^3, \quad J \approx 10^3, \quad R_0 \approx 10^{-5}, \quad L/l \approx 300, \tag{8.2}$$

and so the inequalities

$$Q \gg 1, \quad J \gg 1, \quad R_0 \ll 1, \quad L/l \gg 1. \tag{8.3}$$

are well satisfied. With these values, $Q/J = O(1)$, and the restrictions (5.1) and (5.2) are all equivalent, and are satisfied if the typical period of the disturbing forces is of the order of months or greater. In the absence of knowledge concerning the precise nature of the source of energy for core motions, it is impossible to given any firm estimate for ω_0, but the condition $\omega_0 \ll \Omega$ would seem quite plausible in view of the long time scales associated with most geomagnetic phenomena.

The most important assumption of this paper is that there is some selective mechanism present which leads to a net flux of energy parallel to $\mathbf{\Omega}$; this is in effect the implication of the assumption (2.9); the weaker assumption

$$\langle(\mathbf{A}^{+}(\mathbf{k},\omega))^2\rangle + \langle(\mathbf{B}^{+}(\mathbf{k},\omega))^2\rangle \neq \langle(\mathbf{A}^{-}(\mathbf{k},\omega))^2\rangle + \langle(\mathbf{B}^{-}(\mathbf{k},\omega))^2\rangle \tag{8.4}$$

would undoubtedly result in the same qualitative behaviour. In the earth, it is as yet by no means clear how such an energy flux may arise, but there are various possibilities that would seem to deserve further investigation, *viz.* (i) transmission of energy from incident waves from the core into the weakly conducting mantle; (ii) absorption of energy in the Ekman–Hartmann layer at the core–mantle interface; (iii) the critical-layer absorption mechanism in a sheared magnetic field (Acheson 1972); and (iv) the influence of gradients of mean density and mean turbulence intensity (Steenbeck, Krause & Radler 1966).

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Appendix A. Derivation of the formulae (3.10) and (3.12)

Let $g(\omega)$ be any function, bounded and continuous in $(0, \infty)$. Then the integral

$$I(\lambda) = \int_0^\infty \frac{g(\omega)}{|D|^2} d\omega \tag{A 1}$$

is singular when $\lambda = 0$, since then $|D|^2 = O(\omega - \omega_c)^2$ at the points $\omega = \omega_c$ $(c = 1, 2)$ (equation (3.7)). As $\lambda \to 0$, the asymptotic expansion of $I(\lambda)$ depends only on the behaviour of the integrand near these critical frequencies, and the leading term is given by

$$I(\lambda) \sim \sum_{c=1,2} g(\omega_c) \int_{-\infty}^{\infty} \frac{d\omega}{(\omega-\omega_c)^2 D_{0\omega_c}^2 + \lambda^2 D_{1c}^2}, \tag{A 2}$$

where $D_{0\omega_c}$ and D_{1c} are as defined in (3.8) and (3.9). Hence

$$I(\lambda) \sim \frac{\pi}{\lambda} \sum_{c=1,2} \frac{g(\omega_c)}{D_{0\omega_c} D_{1c}}, \tag{A 3}$$

and substitution of the appropriate functions $g(\omega_c)$ gives the formulae (3.10) and (3.12).

Appendix B. Derivation of the formula (4.12)

From (2.20),
$$\frac{h_{0k}^2}{h_0^2 k^2 \sin^2\theta} = \cos^2\phi = \frac{\omega_c(\omega_c \pm 2\Omega\cos\theta)}{h_0^2 k^2 \sin^2\theta}. \tag{B 1}$$

Hence
$$-\cos\phi\sin\phi\, d\phi = \frac{(\omega_c \pm \Omega\cos\theta)\, d\omega_c}{h_0^2 k^2 \sin^2\theta} = \frac{(h_{0k}^2 + \omega_c^2)\, d\omega_c}{h_0^2 k^2 \sin^2\theta}. \tag{B 2}$$

Substitution in (4.11) (with due care in the choice of signs) leads directly to (4.12) and (4.13). The range of integration for μ in (4.13) is determined from (B 1) by the double requirement
$$1 \pm p\mu \geqslant 0, \quad 1-\mu^2 \geqslant q^2(1 \pm p\mu). \tag{B 3}$$

The appropriate interval of the μ axis in which both these inequalities are satisfied is indicated in figure 1, for $c = 1, 2$, and for various representative values of the parameters p and q.

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